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Reduction of the codimension for lightlike isotropic submanifolds

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Abstract

We give a sufficient condition for a lightlike isotropic submanifold M, of dimension n, which is not totally geodesic in a semi-Riemannian manifold of constant curvature c and of dimension $n + p(n < p)$, to admit a reduction of codimension. We show that this condition is a necessary and sufficient condition on the first transversal space of M. There are basic and non-trivial differences from the Riemannian case, as developed by Dajczer et al. [Mathematics Lectures Series, Vol. 13, 1990], due to the degenerate metric on M. This result extends in some sense, the one in [J. Diff. Geom. 5 (1971) 333; Topology 25 (4) (1986) 541; Mathematics Lectures Series, Vol. 13, 1990] to lightlike isotropic submanifolds.

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1. Introduction

A natural generalization of the pioneering work by Gauss in differential geometry was the study of submanifolds $f : M^n \to \mathbb{R}^{n+p}$, of arbitrary codimension p immersed into Euclidean $(n + p)$-spaces. An extensive work has been devoted to these submanifolds and many results are now referred to as classical ones on their geometrical structure. Mainly the case in which the induced metrics on M are non-degenerate are examined for instance in [3,5–7] and references therein.

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In a recent past, the growing importance of lightlike submanifolds in global Lorentzian geometry and their use in general relativity, motivated the study of degenerated submanifolds in a semi-Riemannian manifold. Due to the degeneracy of the metric, basic differences occur between the study of lightlike submanifolds and the classical theory of Riemannian as well as semi-Riemannian submanifolds [4,9,11].

In a point of view of physics, the idea that the universe we live in can be represented as a four-dimensional hypersurface embedded in a $(4 + d)$-dimensional space–time manifold has attracted the attention of many physicists. The embedding of exact solutions of Einstein equations into higher dimensional semi-Euclidean space can give a more adequate picture and a better understanding of their intrinsic geometry. Higher dimensional semi-Euclidean spaces should provide theoretical framework in which the fundamental laws of physics may appear to be unified, as in the Kaluza–Klein scheme, which takes into account the mutual interaction between matter and metric [9,10].

From the point of view of mathematics, methods and results of submanifolds study in differential geometry might be revisited with a greater interest to the context of degeneracy. Sometimes they drastically change from non-degenerate metric case to the degenerate metric one. As far as we know a few literature is available on the theory of lightlike submanifolds in semi-Riemannian manifolds. The basic work seems to be the series by Duggal and Bejancu [4] and also Kupeli [8].

In this paper, generalizing earlier results in [1–3], we give sufficient condition for a lightlike isotropic submanifold of dimension n, which is not totally geodesic in a semi-Riemannian manifold of constant curvature c and of dimension $n + p(n < p)$, to admit a reduction of codimension, i.e. to be immersed into an $(n+q)$-dimensional totally geodesic submanifold of constant curvature, with $q < p$. Our main results stand as follows.

Theorem 1. *Let $f : M^n \to \bar{M}_c^{n+p}$ be an isometric immersion of an isotropic submanifold $(M, g, S(TM^\perp))$ into a complete and simply connected semi-Riemannian manifold with constant sectional curvature c $(\bar{M}_c^{n+p}, \bar{g})$. Suppose that*:

1. *The transversal connection ∇^{t} on M^n is metric.*
2. *There exists a screen transversal subbundle P of $S(TM^\perp)$ of constant rank q $(q < p)$, parallel w.r.t. the connection ∇^{s} on $S(TM^\perp)$, such that*

$$T_1(x) \subset P(x) \quad \forall x \in M,$$

where $T_1(x)$ is the first transversal space of f at $x \in M$.

Then the codimension of f can be reduced to q.

The isometric immersion f is said to be 1-regular if the dimension of the transversal space is constant along M, and this notion is independent of the metric of M. In this case, the substantial codimension [3, p. 54], or the embedding class of M [9,11] is the lowest value of q. We show that the substantial codimension of M^n is equal to the rank of its first transversal space $T_1(x)$ when the latter is of constant rank q_0 on M^n. We have the following theorem.

Theorem 2. *Let $(M^n, g, S(TM^\perp))$ be an isometric immersion of an isotropic non-totally geodesic submanifold in $\bar{M}_c^{n+p}(n < p)$. Then the subbundle T_1 is parallel w.r.t. the connection ∇^s on $S(TM^\perp)$.*

The paper is organized as followed. In Section 1, we summarize notations and basic formulas concerning geometric objects on lightlike submanifolds, using notations of [4]. Section 2 gives the set up necessary for the proof of the theorems and Section 3 gives the proofs. Appendix A shows a motivating example to illustrate the purpose of the paper.

2. Preliminaries and basic facts

2.1. *The general set up*

The fundamental difference between the theory of lightlike (or degenerate) submanifolds (M^n, g), and the classical theory of submanifolds of a semi-Riemannian manifold $(\bar{M}^{n+p}, \bar{g})$ comes from the fact that in the first case, the normal vector bundle $TM^\perp$ intersects with the tangent bundle TM in a non-zero subbundle, denoted $\mathrm{Rad}(TM)$, so that

$$\mathrm{Rad}(TM) = TM \cap TM^\perp \neq \{0\}. \tag{1}$$

Given an integer $r > 0$, the submanifold M is said to be r-lightlike if $\mathrm{rank}(\mathrm{Rad}(TM)) = r$ everywhere.

An orthogonal complementary vector subbundle of $\mathrm{Rad}(TM)$ in TM is a non-degenerate subbundle of TM called a screen distribution on M and denoted $S(TM)$. We have the following splitting into an orthogonal direct sum:

$$TM = S(TM) \perp \mathrm{Rad}(TM). \tag{2}$$

From Eq. (1), we can consider a complementary vector subbundle $S(TM^\perp)$ of $\mathrm{Rad}(TM)$ in $TM^\perp$. It is also a non-degenerate subbundle with respect to the metric $\bar{g}$, and we have

$$TM^\perp = \mathrm{Rad}(TM) \perp S(TM^\perp). \tag{3}$$

The subbundle $S(TM^\perp)$ is a screen transversal vector bundle of M. The subbundle $S(TM)$ being non-degenerate, so is $(S(TM))^\perp$ and the following holds:

$$T\bar{M}|_M = S(TM) \perp (S(TM))^\perp. \tag{4}$$

Note that $S(TM^\perp)$ is a subbundle of $(S(TM))^\perp$ and, since both are non-degenerate, we have

$$(S(TM))^\perp = S(TM^\perp) \perp (S(TM^\perp))^\perp. \tag{5}$$

One frequently denotes a lightlike submanifold M by $(M, S(TM), S(TM^\perp))$ to refer to the above subbundles.

In fact, $\mathrm{Rad}(TM)$ is a subbundle of $(S(TM^\perp))^\perp$. Let $\mathrm{ltr}(TM)$ denote its complementary vector bundle in $(S(TM^\perp))^\perp$. One has

$$(S(TM^\perp))^\perp = \mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM).$$

The subbundle $\mathrm{ltr}(TM)$ is called a lightlike transversal vector bundle of M. The subbundle $\mathrm{tr}(TM)$ defined by

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp})$$

is called a transversal vector bundle of M and plays an important role in the study of the geometry of lightlike submanifolds. We always have $\mathrm{tr}(TM) \cap TM^{\perp} \neq \mathrm{tr}(TM)$. That is $\mathrm{tr}(TM)$ is never orthogonal to TM. From now on, given a vector bundle E, we denote $\Gamma(E)$ the space of smooth sections of E.

Summarizing the above statements, we have the following decomposition:

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM) = S(TM) \perp S(TM^{\perp}) \perp (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)), \tag{6}$$

which gives rise to a local quasi-orthonormal field of frames on $\bar{M}$ along M (see [4]) denoted by $(\xi_i, N_i, X_a, W_\alpha)$, where

1. $\{\xi_i\}$ and $\{N_i\}$, $i \in \{1, \dots, r\}$ are lightlike basis of $\Gamma(\mathrm{Rad}(TM)|_{\mathcal{U}})$ and $\Gamma(\mathrm{ltr}(TM)|_{\mathcal{U}})$, respectively,
2. $\{X_a\}$, $a \in \{r+1, \dots, m\}$ is an orthonormal basis of $\Gamma(S(TM)|_{\mathcal{U}})$,
3. $\{W_\alpha\}$, $\alpha \in \{r+1, \dots, n\}$ an orthonormal basis of $\Gamma(S(TM^{\perp})|_{\mathcal{U}})$,

relative to a coordinate neighborhood $\mathcal{U} \subset M$.

A lightlike submanifold is said to be *isotropic* if $\mathrm{Rad}(TM) = TM$. In this case, we deduce from (2) that $S(TM) = \{0\}$. This requires that $n < p$ and the formula (6) reduces to

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM) = S(TM^{\perp}) \perp (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)). \tag{7}$$

In the sequel, the lightlike submanifold M is supposed to be isotropic.

2.2. Induced connections

Let $\bar{\nabla}$ denoted the Levi-Civita connection on $\bar{M}$ and ∇ the induced connection on M. For all $X, Y \in \Gamma(TM)$, and $V \in \Gamma(\mathrm{tr}(TM))$, we deduce from (7) that

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\mathrm{l}}(X, Y) + h^{\mathrm{s}}(X, Y) \tag{8}$$

and

$$\bar{\nabla}_X V = -A_V X + D^{\mathrm{l}}_X V + D^{\mathrm{s}}_X V, \tag{9}$$

where h^{l} and h^{s} are $\Gamma(\mathrm{ltr}(TM))$-valued, and $\Gamma(S(TM^{\perp}))$-valued, respectively. They are called the lightlike and the screen second fundamental forms of M, respectively. As usual, A_V denotes the shape operator with respect to V.

The second fundamental form of M with respect to $\mathrm{tr}(TM)$ is defined by

$$h(X, Y) = h^{\mathrm{l}}(X, Y) + h^{\mathrm{s}}(X, Y), \quad X, Y \in \Gamma(TM). \tag{10}$$

Let L and S denote the projection morphism of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively. In (9) we have

$$D^{\mathrm{l}}_X V = L(\nabla^{\mathrm{t}}_X V), \quad D^{\mathrm{s}}_X V = S(\nabla^{\mathrm{t}}_X V) \quad \forall X \in \Gamma(TM)\ \forall V \in \Gamma(\mathrm{tr}(TM)),$$

where ∇^{t}_X stands for the transversal linear connection on M. The transformations D^{l} and D^{s} do not define linear connections on $\mathrm{tr}(TM)$ [4, p. 27], but define two Otsuki connections on $\mathrm{tr}(TM)$ with respect to the vector bundle morphisms L and S.

Since the submanifold M is isotropic, the lightlike second fundamental form h^{l} vanishes identically on M [4, p. 157].

Define the $C^{\infty}(M)$-bilinear mappings, D^{l} and D^{s} by

$$D^{\mathrm{l}} : \Gamma(TM) \times \Gamma(S(TM^{\perp})) \to \Gamma(\mathrm{ltr}(TM)), \qquad (X, SV) \mapsto D^{\mathrm{l}}(X, SV) = D^{\mathrm{l}}_X(SV)$$

and

$$D^{\mathrm{s}} : \Gamma(TM) \times \Gamma(\mathrm{ltr}(TM)) \to \Gamma(S(TM^{\perp})), \qquad (X, LV) \mapsto D^{\mathrm{s}}(X, LV) = D^{\mathrm{s}}_X(LV).$$

Then we have

$$\bar{\nabla}_X N = -A_N X + \nabla^{\mathrm{l}}_X N + D^{\mathrm{s}}(X, N), \tag{11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^{\mathrm{s}}_X W + D^{\mathrm{l}}(X, W), \tag{12}$$

where ∇^{s} and ∇^{l} are linear connections on $S(TM^{\perp})$ and $\mathrm{ltr}(TM)$, respectively; $X \in \Gamma(TM)$, $N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

As shown in [4, p. 166] M is totally geodesic if and only if $D^{\mathrm{l}}(\cdot, W) = 0$ for all $W \in \Gamma(S(TM^{\perp}))$.

A direct computation shows that, for all $X \in \Gamma(TM)$, $V, V' \in \Gamma(\mathrm{tr}(TM))$ we have

$$(\nabla^{\mathrm{t}}_X \bar{g})(V, V') = -(\bar{g}(A_V X, V') + \bar{g}(A_{V'} X, V)), \tag{13}$$

so that the transversal linear connection ∇^{t} on $\mathrm{tr}(TM)$ is not metric in general.

The first transversal space at $x \in M$ of the isometric immersion f is defined as the subspace

$$T_1(x) = \mathrm{span}\{h^{\mathrm{s}}(X, Y), X, Y \in \Gamma(T_x M)\}.$$

For the proof of theorems, we need the following two lemmas.

Lemma 1. *If the transversal linear connection ∇^{t} on $\mathrm{tr}(TM)$ is metric, then $A_W = 0$ for all $W \in \Gamma(S(TM^{\perp}))$.*

Proof. Due to Eq. (13), ∇^{t} is metric, if and only if A_W is $\Gamma(S(TM))$-valued for all $W \in \Gamma(S(TM^{\perp}))$. The lemma follows from the fact that M being isotropic, $S(TM) = \{0\}$. □

Lemma 2. *For any $x \in M$, the first transversal space $T_1(x)$ has the characterization,*

$$T_1(x) = \{V = W + N \in \Gamma(\mathrm{tr}(TM)), D^{\mathrm{l}}(\cdot, W) = 0\}^{\perp}. \tag{14}$$

Proof. Because M is a non-totally geodesic isotropic submanifold of $\bar{M}$, Lemma 2 shows that T_1 is not trivial, that is $T_1(x) \neq \{0\}$, for all $x \in M$.

Let $V = h^{\mathrm{s}}(X, Y)$, $X, Y \in \Gamma(TM)$, be a generic element of $T_1(x)$ and $U \in A(x)^{\perp}$ with

$$A(x) := \{V = W + N \in \Gamma(\mathrm{tr}(TM)),\ D^{\mathrm{l}}(\cdot, W) = 0\}^{\perp},$$

then

$$\bar{g}(U, V) = \bar{g}(h^{\mathrm{s}}(X, Y), W + N) = g(A_W X, Y) - \bar{g}(Y, D^{\mathrm{l}}(X, W)) = 0,$$

where we use Lemma 1 and the definition of $A(x)$. Thus,

$$V \in T_1(x) \Leftrightarrow \bar{g}(V, U) = 0 \quad \forall U \in A(x)^{\perp} \Leftrightarrow V \in (A(x)^{\perp})^{\perp} = A(x),$$

so $T_1(x) = A(x)$. □

3. Proof of theorems

3.1. *Proof of Theorem 1*

First of all, note that P is a ∇^{s}-parallel subbundle of constant rank q of the bundle $S(TM^{\perp})$ implies that

$$\nabla^{\mathrm{s}}_X W \in P \quad \forall X \in \Gamma(TM)\ \ \forall W \in \Gamma(P).$$

Then consider as usual the three cases $c = 0$, $c > 0$ and $c < 0$.

3.1.1. *Case* $c = 0$

For $x_0 \in M$, we prove that $f(M) \subset T_{x_0}M \oplus P(x_0)$. Let μ be a section of the complementary orthogonal bundle of P in $S(TM^{\perp})$, $\gamma : I \to M$ a regular curve on M and μ_t the parallel transport of μ along γ.

Since P is parallel in $\Gamma(S(TM^{\perp}))$, so is its orthogonal complementary $P^{\perp}$ in the subbundle $\Gamma(S(TM^{\perp}))$ and

$$\mu_t = \nabla^{\mathrm{s}}_{\gamma'}\mu \in \Gamma(P^{\perp}_{\gamma(t)}) \quad \forall t \in I.$$

Using Weingarten formula, we have

$$\bar{\nabla}_{\gamma'}\mu_t = -A_{\mu_t}\gamma' + D^{\mathrm{l}}(\gamma', \mu_t) + \nabla^{\mathrm{s}}_{\gamma'}\mu_t.$$

But

$$\mu_t \in \Gamma(P^{\perp}_{\gamma(t)}) \subset \Gamma(S(TM^{\perp})) \quad \forall t \in I.$$

Lemma 1 yields $A^{\gamma'}_{\mu_t} = 0$ for all $t \in I$.

Moreover, $\mu_t \in P^{\perp}_{\gamma(t)} \subset T_1(\gamma(t))$ from assumption of the theorem. So Lemma 2 gives

$$D^{\mathrm{l}}(\gamma', \mu_t) = 0 \quad \forall\, t \in I.$$

And because μ_t is the parallel transport in $P^{\perp}$ of μ along γ, we have $\nabla^{\mathrm{s}}_{\gamma'}\mu_t = 0$ for all t in I.

We deduce that $\bar{\nabla}_{\gamma'}\mu_t = 0$ for all $t \in I$, so that $\mu_t = \mu$ is a constant vector in $\mathbb{R}^{n+p}$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(f(\gamma(t)) - f(x_0), \mu_t) = \bar{g}(f_*\gamma'(t), \mu) = 0.$$

We conclude that

$$\bar{g}(f(\gamma(t)) - f(x_0), \mu) = 0 \quad \forall t \in I$$

and

$$f(\gamma(t)) - f(x_0) \in (P^{\perp}_{\gamma(t)})^{\perp} = P_{\gamma(t)} \quad \forall t \in I.$$

Due to the fact that γ and μ are arbitrary on M, we have

$$f(M) \subset T_{x_0}(M) \oplus P(x_0) \cong \mathbb{R}^{n+q},$$

which is a totally geodesic $(n+q)$-dimensional subspace of $\mathbb{R}^{n+p}$.

3.1.2. Case $c > 0$

The isotropic submanifold M^n is isometrically immersed into a pseudosphere S_c^{n+p}. Consider the isometric immersion,

$$\tilde{f} = i \circ f : M^n \to \mathbb{R}^{n+p+1},$$

where the map i is the natural injection of S_c^{n+p} into $\mathbb{R}^{n+p+1}$. Then

$$\mathrm{tr}(\tilde{T}_x M) = \mathrm{tr}(T_x M) \oplus \langle f(x)\rangle$$

with

$$\langle f(x)\rangle \subset S(\tilde{T}_x M^{\perp}),$$

where $\langle f(x)\rangle := \mathrm{Span}\{f(x)\}$.

We deduce that

$$\tilde{T}_1(x) \subset T_1(x) \oplus \langle f(x)\rangle \subset P(x) \oplus \langle f(x)\rangle = \tilde{P}(x).$$

And then

$$\tilde{T}_1(x) \subset S(T_x M^{\perp}) \oplus \langle f(x)\rangle = S(\tilde{T}M^{\perp}) \quad \forall x \in M.$$

The orthogonal complementary of $\tilde{P}(x)$ in $S(\tilde{T}M^{\perp})$ and of $P(x)$ in $S(TM^{\perp})$, which is parallel w.r.t. the transversal screen connection $\nabla^{\mathrm{s}} = \tilde{\nabla}^{\mathrm{s}}|_{S(TM^{\perp})}$, are equal, and

$$\tilde{\nabla}^{\mathrm{s}}_X W = \tilde{D}^{\mathrm{s}}_X W = D^{\mathrm{s}}(X, W) \quad \forall W \in \Gamma(\tilde{T}(S(TM^{\perp}))).$$

Thus

$$\bar{g}(\tilde{\nabla}^{\mathrm{s}}_X \tilde{f}(x), W) = \bar{g}((\bar{D}^{\mathrm{s}}_X \tilde{f}(x), W) = X\bar{g}(\tilde{f}(x), W) - \bar{g}(\tilde{f}(x), \nabla^{\mathrm{s}}_X W) = 0$$

and therefore

$$\tilde{\nabla}^{\mathrm{s}}_X \tilde{f}(x) \in \langle f(x) \rangle$$

and $\langle f(x) \rangle$ is a transversal vector subbundle who is parallel w.r.t. the connection $\tilde{\nabla}^{\mathrm{s}}$. We conclude that $\tilde{P}$ is parallel w.r.t. $\tilde{\nabla}^{\mathrm{s}}$, and as in the case $c = 0$, we have

$$\tilde{f}(M) \subset \tilde{T}_{x_0} M \oplus \tilde{P}(x_0) = T_{x_0}(M) \oplus P(x_0) \oplus \langle f(x_0) \rangle \cong \mathbb{R}^{n+q+1}.$$

So $f(M) \subset S_c^{n+p} \cap \mathbb{R}^{n+q+1} = S_c^{n+q}$ which is totally geodesic in S_c^{n+p}. This proves the case $c > 0$.

3.1.3. *Case $c < 0$*

The general scheme holds as for $c > 0$. Now $\tilde{f}$ maps M^n into $\mathbb{L}^{n+p+1}$, the Lorentzian space $\mathbb{R}_1^{n+p+1}$ and we get that

$$\tilde{f}(M) \subset \tilde{T}_{x_0} M \oplus \tilde{P}(x_0) = T_{x_0}(M) \oplus P(x_0) \oplus \langle f(x_0) \rangle,$$

where $f(x)$ is spacelike. Then

$$\tilde{f}(M) \subset \mathbb{L}^{n+q+1}$$

and

$$f(M) \subset \mathbb{H}_c^{n+p} \cap \mathbb{L}^{n+q+1} \cong \mathbb{H}_c^{n+q}$$

and M admits a reduction of codimension, which completes the proof.

3.2. *Proof of Theorem 2*

We have $T_1(x) \subset (S(TM^\perp))\ \forall x \in M$. To prove that T_1 is parallel, we will prove that its orthogonal complementary in $\Gamma(S(TM^\perp))$ is parallel. So, if $\eta \in T_1^\perp$, we have to prove that

$$\nabla^{\mathrm{s}}_Z \eta \in T_1^\perp \quad \forall Z \in \Gamma(TM),$$

i.e.

$$\bar{g}(h^{\mathrm{s}}(X, Y), \nabla^{\mathrm{s}}_Z \eta) = 0 \quad \forall X, Y, Z \in \Gamma(TM). \tag{15}$$

Set

$$\eta = N + W, \quad N \in \Gamma(\mathrm{ltr}(TM)), \qquad W \in \Gamma(S(TM^\perp)),$$

then

$$\bar{g}(h^{\mathrm{s}}(X, Y), \nabla^{\mathrm{s}}_Z \eta) = g(A_{\nabla^{\mathrm{s}}_Z} \eta, Y) - \bar{g}(Y, D^{\mathrm{l}}(X, W)).$$

But using Lemma 1, we have

$$\nabla^{\mathrm{s}}_Z \eta \in \Gamma(S(TM^\perp)) \Rightarrow A_{\nabla^{\mathrm{s}}_Z \eta} = 0$$

and

$$\eta = N + W \in T_1^{\perp} \Rightarrow D^{\mathrm{l}}(X, W) = 0.$$

We deduce that

$$\bar{g}(h^{\mathrm{s}}(X, Y), \nabla^{\mathrm{s}}_Z \eta) = 0 \quad \forall X, Y, Z \in \Gamma(TM)$$

and then $\nabla^{\mathrm{s}}_Z \eta \in T_1^{\perp}$ so that $N_1^{\perp}$ is parallel w.r.t. the connection ∇^{s}. This proves Theorem 2.

As a consequence of the two theorems, we have the following:

Proposition 1. *A necessary and sufficient condition for the isotropic immersion $f : M^n \to \tilde{M}_c^{n+p}$, $n < p$ to admit a reduction of codimension, is that the isotropic immersion is 1-regular of constant rank q, and the substantial codimension is q.*

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Appendix A

These ideas are illustrated through the following example.

Suppose M is a surface of $\mathbb{R}^5_2$, Euclidean space $\mathbb{R}^5$ with a semi-Euclidean metric $\bar{g} = \mathrm{diag}(-1, -1, +1, +1, +1)$, given by equations,

$$x^1 = \frac{1}{\sqrt{2}}(x^4 + \sinh x^5), \qquad x^2 = \frac{1}{\sqrt{2}}(x^4 - \sinh x^5), \quad x^3 = \cosh x^5$$

and set $(u = x^4, v = x^5)$ a system of coordinate on M. We derive the following:

$$TM = \mathrm{Span}\{\xi_1, \xi_2\}$$

with

$$\xi_1 = \frac{\partial}{\partial u} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4},$$

$$\xi_2 = \frac{\partial}{\partial v} = \frac{\cosh x^5}{\sqrt{2}}\frac{\partial}{\partial x^1} - \frac{\cosh x^5}{\sqrt{2}}\frac{\partial}{\partial x^2} + \sinh x^5 \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5}$$

and

$$TM^{\perp} = \mathrm{Span}\left\{U_1 = \xi_1, U_2 = \xi_2, U_3 = \frac{\partial}{\partial x^3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^4} - \sinh x^5 \frac{\partial}{\partial x^5}\right\}.$$

It follows that $\mathrm{Rad}(TM) = TM \subset TM^{\perp}$ and M is an isotropic surface of $\mathbb{R}^5_2$.

The subbundle $S(TM^{\perp})$ is a complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$. We take (there is no unicity),

$$S(TM^{\perp}) = \mathrm{Span}\left\{W_1 = \frac{\sinh x^5}{\sqrt{2}}\frac{\partial}{\partial x^1} - \frac{\sinh x^5}{\sqrt{2}}\frac{\partial}{\partial x^2} + \cosh x^5 \frac{\partial}{\partial x^3}\right\}.$$

A.1. Construction of $\mathrm{ltr}(TM)$

A basis $\{N_1, N_2\}$ of $\mathrm{ltr}(TM)$ on a coordinate neighborhood $\mathcal{U}$ satisfies:

$$\bar{g}(N_i, N_j) = 0 \quad \forall i, j \in \{1, 2\}, \qquad \bar{g}(\xi_1, N_2) = \bar{g}(\xi_2, N_1) = 0,$$
$$\bar{g}(N_1, \xi_1) = \bar{g}(N_2, \xi_2) = 1. \tag{A.1}$$

Using (A.1) we obtain that

$$\mathrm{ltr}(TM) = \mathrm{Span}\{N_1, N_2\}$$

with

$$N_1 = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\frac{\partial}{\partial x^1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4}\right),$$
$$N_2 = \frac{1}{2}\left(-\frac{\cosh x^5}{\sqrt{2}}\frac{\partial}{\partial x^1} + \frac{\cosh x^5}{\sqrt{2}}\frac{\partial}{\partial x^2} - \sinh x^5 \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5}\right)$$

and deduce that

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \oplus \mathrm{Rad}(TM) = \mathrm{Span}\{W_1, N_1, N_2\}.$$

A straightforward calculation gives

$$\bar{\nabla}_{\xi_1}\xi_1 = \bar{\nabla}_{\xi_1}\xi_2 = \bar{\nabla}_{\xi_2}\xi_1 = 0, \qquad \bar{\nabla}_{\xi_2}\xi_2 = W_1.$$

We deduce that M is not totally geodesic in $\mathbb{R}^5_2$.

Moreover we have for all $X, Y, \in \Gamma(TM)$, $X = X^i\xi_i$, $Y = Y^j\xi_j$,

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\mathrm{s}}(X, Y)$$
$$= [(X^1(\xi_1(Y^1)) + X^2(\xi_2(Y^1)))\xi_1 + (X^1(\xi_1(Y^2)) + X^2(\xi_2(Y^2)))\xi_2] + X^2Y^2W_1$$

and

$$h^{\mathrm{s}}(X, Y) = \bar{g}(X, N_2)\bar{g}(Y, N_2)W_1.$$

So that

$$h_1^{\mathrm{s}}(\xi_2, \xi_2) = 1. \tag{A.2}$$

From (A.2) we infer that

$$T_1(x) = \mathrm{Span}\{h^{\mathrm{s}}(X, Y), X, Y \in \Gamma(T_xM)\} = S(T_xM^{\perp}),$$

which is of constant rank $q = 1$ for all $x \in M$. From above and proposition, M admits a reduction of codimension to 1, that is there exists a totally geodesic three-dimensional submanifold of $\mathbb{R}_2^5$ into which M can be isometrically immersed.

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